

# Static Solutions with Spherical Symmetry in $f(T)$ Theories

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(Dated: January 12, 2013)

## Abstract

The spherically symmetric static solutions are searched for in some  $f(T)$  models of gravity theory with a Maxwell term. To do this, we demonstrate that reconstructing the Lagrangian of  $f(T)$  theories is sensitive to the choice of frame, and then we introduce a particular frame based on the conformally Cartesian coordinates. In this particular frame, the existence conditions of various solutions are presented. Our results imply that only a limited class of  $f(T)$  models can be solved in this frame. For more general models, the search for spherically symmetric static solutions is still an open and challenging problem, hopefully solvable in other frames.

PACS numbers: 04.50.Kd, 04.20.Jb

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## I. INTRODUCTION AND PRELIMINARIES

As a great triumph of science in the last century, Einstein's theory of general relativity with a cosmological constant has been established by many observations and experiments hitherto. In spite of this fact, people have been enthusiastically trying to modify or extend this theory. Cosmologically the motivation to do it is running our Universe with less dark matter [1, 2] or without the cosmological constant [3–5], or driving inflation without a scalar field [6, 7] or with richer phenomena [8].

A few years ago,  $f(T)$  theories of gravity were proposed as an alternative of the cosmological constant to explain the accelerated expansion of the late Universe [9]. Similar to the  $f(R)$  theories,  $f(T)$  theories deviate from Einstein gravity by a function  $f(T)$  in the Lagrangian, where  $T$  is the so-called torsion scalar. This class of theories has received a lot of attention recently [10–30], but earlier examples could be traced to [31, 32]. In order to better study  $f(T)$  gravity theories, we will seek for exact solutions of the field equations, akin to the work [33–37] done in  $f(R)$  models. Especially we are interested in spherically symmetric solutions. The widely studied Friedmann-Lemaître-Robertson-Walker (FLRW) metric is such a solution obviously, but we will pay attention to static solutions instead.

In this paper, the notation of indices is as follows: Greek indices  $\mu, \nu, \dots$  and Latin indices from the beginning of the alphabet  $a, b, \dots$  run over 0, 1, 2, 3, while Latin indices from the middle of the alphabet  $i, j, \dots$  run from 1 to 3. In our notations, the Kronecker delta is defined with the same normalization irrespective of the position of indices, *e.g.*  $\delta_{00} = \delta^{00} = \delta_0^0 = 1$ .

The dynamic variables in  $f(T)$  theories are the tetrad (vierbein) fields  $\mathbf{e}_a$ . In a coordinate basis, the components of dual tetrad fields  $e_\mu^a dx^\mu$  are related to the familiar metric tensor  $g_{\mu\nu}$  through

$$g_{\mu\nu} = \eta_{ab} e_\mu^a e_\nu^b, \quad (1)$$

where  $\eta_{ab} = \text{diag}(1, -1, -1, -1)$ . Starting from the tetrad fields, one may follow [9, 17, 24] to construct the torsion tensor

$$T^\lambda_{\mu\nu} = e_a^\lambda (\partial_\mu e_\nu^a - \partial_\nu e_\mu^a), \quad (2)$$

the contorsion tensor

$$K^{\mu\nu}{}_\lambda = -\frac{1}{2}(T^{\mu\nu}{}_\lambda - T^{\nu\mu}{}_\lambda - T_\lambda{}^{\mu\nu}) \quad (3)$$

and a useful tensor

$$S_\lambda{}^{\mu\nu} = \frac{1}{2}(K^{\mu\nu}{}_\lambda + \delta_\lambda^\mu T^{\rho\nu}{}_\rho - \delta_\lambda^\nu T^{\rho\mu}{}_\rho). \quad (4)$$

Then in terms of the torsion scalar

$$T = S_\lambda{}^{\mu\nu} T^\lambda_{\mu\nu} \quad (5)$$

and the notation  $e = \det(e_\mu^a) = \sqrt{-g}$ , the action of  $f(T)$  gravity theories is given by

$$I = \int d^4x e \left[ \frac{1}{16\pi G} (T + f) + \mathcal{L}_m \right], \quad (6)$$

which leads to the equations of motion

$$(1 + f_{,T})[e^{-1}\partial_\mu(eS_a{}^{\mu\nu}) - e_a^\lambda T^\rho_{\mu\lambda} S_\rho{}^{\nu\mu}] + f_{,TT} S_a{}^{\mu\nu} \partial_\mu T - \frac{1}{4} e_a^\nu (T + f) = 4\pi G e_a^\rho \mathcal{T}_\rho{}^\nu \quad (7)$$

with  $S_a^{\mu\nu} = e_a^\rho S_\rho^{\mu\nu}$ . If the function  $f(T)$  is replaced by a constant, the theory is equivalent to Einstein's theory with a cosmological constant [24].

Throughout this paper, the Maxwell term

$$\mathcal{L}_m = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} \quad (8)$$

will be considered. It gives the energy-momentum tensor

$$\mathcal{T}_\rho^\nu = F_{\rho\lambda}F^{\nu\lambda} - \frac{1}{4}\delta_\rho^\nu F_{\lambda\mu}F^{\lambda\mu}. \quad (9)$$

We will need the Ricci scalar  $R = g^{\mu\nu}R_{\mu\nu}$ , taking the convention of Ricci tensor

$$R_{\mu\nu} = \partial_\lambda \Gamma_{\mu\nu}^\lambda - \partial_\nu \Gamma_{\mu\lambda}^\lambda + \Gamma_{\lambda\rho}^\lambda \Gamma_{\mu\nu}^\rho - \Gamma_{\nu\rho}^\lambda \Gamma_{\mu\lambda}^\rho \quad (10)$$

with the Levi-Civita connection

$$\Gamma_{\mu\nu}^\lambda = \frac{1}{2}g^{\lambda\rho}(\partial_\mu g_{\rho\nu} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu}). \quad (11)$$

In the rest of this paper, after choosing the frame in section II, we will search for the spherically symmetric static solutions in different situations in sections III, IV, V. Then we will recap the solutions and their constraints on Lagrangian in section VI. In section VII some open issues will be mentioned.

## II. THE CHOICE OF FRAME

The first difficulty we encountered is the choice of frame. As dynamic variables in  $f(T)$  theories, tetrad fields are sensitive to the frame. More importantly, the torsion scalar is also frame-sensitive, very much unlike the Ricci scalar. This trouble is attributed to the lack of local Lorentz invariance in  $f(T)$  theories [24]. Consequently, even for the same metric and the same coordinate basis, different frames result in different forms of equations of motion.

To see this point clearly, we take the FLRW metric for example. Corresponding to the coordinates

$$ds^2 = dt^2 - a^2(t)\delta_{ij}dx^i dx^j, \quad (12)$$

the following form of tetrads are widely used in the literature [9, 10],

$$e_\mu^0 dx^\mu = \delta_\mu^0 dx^\mu, \quad e_\mu^i dx^\mu = a\delta_\mu^i dx^\mu. \quad (13)$$

It give the torsion scalar  $T = -6H^2$  and the Ricci scalar  $R = -6(2H^2 + \dot{H})$ . Here the Hubble parameter  $H = \dot{a}/a$ , and the dot overhead implies the derivative with respect to  $t$ . Then equations of motion (7) lead to the generalized Friedmann equations

$$\begin{aligned} f + 6H^2 + 12H^2 f_{,T} &= 16\pi G\rho, \\ f + 6H^2 + 4\dot{H} + 4(3H^2 + \dot{H})f_{,T} - 48H^2 \dot{H} f_{,TT} &= -16\pi Gp. \end{aligned} \quad (14)$$

Expression (13) is an obvious form of tetrads yielding metric (12). But it is not the unique choice. For instance, we can rotate it to a different frame, and write down the following

form of tetrads:

$$\begin{aligned}
e_\mu^0 dx^\mu &= dt, \\
e_\mu^1 dx^\mu &= \frac{a}{r}(x dx + y dy + z dz), \\
e_\mu^2 dx^\mu &= \frac{az}{r\sqrt{x^2+y^2}}(x dx + y dy) - \frac{a\sqrt{x^2+y^2}}{r} dz, \\
e_\mu^3 dx^\mu &= -\frac{a}{\sqrt{x^2+y^2}}(y dx - x dy),
\end{aligned} \tag{15}$$

in which  $r = (x^2 + y^2 + z^2)^{1/2}$ . It is easy to check that the new form of tetrads can also produce metric (12) and the local rotation matrix

$$\begin{pmatrix} \frac{x}{r} & \frac{y}{r} & \frac{z}{r} \\ \frac{xz}{r\sqrt{x^2+y^2}} & \frac{yz}{r\sqrt{x^2+y^2}} & -\frac{\sqrt{x^2+y^2}}{r} \\ -\frac{y}{\sqrt{x^2+y^2}} & \frac{x}{\sqrt{x^2+y^2}} & 0 \end{pmatrix} \tag{16}$$

transforms (13) to (15). Keep in mind that the rotation group is a subgroup of the proper Lorentz transformation, so frames (13) and (15) are related by a local Lorentz transformation. In accordance with tetrads (15), the torsion scalar becomes  $T = 2a^{-2}r^{-2} - 6H^2$  but the Ricci scalar remains  $R = -6(2H^2 + \dot{H})$ . Now equations of motion (7) reduce to

$$\begin{aligned}
f + 6H^2 + 12H^2 f_{,T} - \frac{2}{a^2 r^2} f_{,T} &= 16\pi G\rho, \\
f_{,TT} &= 0, \\
f + 6H^2 + 4\dot{H} + 4(3H^2 + \dot{H})f_{,T} - \frac{2}{a^2 r^2} f_{,T} &= -16\pi Gp.
\end{aligned} \tag{17}$$

Comparing (14) and (17), we can see the equations of motion, as well as the reconstruction of  $f(T)$ , is quite sensitive to the choice of frame. Especially, the second equation of (17) requires  $f = \lambda T - 2\Lambda$ . After rescaling the Newtonian constant, the resulted theory is simply equivalent to Einstein's theory with a cosmological constant. But the commonly studied equations (14), which are derived in frame (13), allow much more general forms of  $f(T)$  to survive. This property does not rely on the choice of coordinate system<sup>1</sup>. Through the coordinate transformation  $x = r \sin \theta \cos \phi$ ,  $y = r \sin \theta \sin \phi$ ,  $z = r \cos \theta$ , expression (15) is traded to

$$e_\mu^0 dx^\mu = dt, \quad e_\mu^1 dx^\mu = a dr, \quad e_\mu^2 dx^\mu = a r d\theta, \quad e_\mu^3 dx^\mu = a r \sin \theta d\phi \tag{18}$$

in a spherical coordinate system

$$ds^2 = dt^2 - a^2(t)(dr^2 + r^2 d\Omega^2) \tag{19}$$

with  $d\Omega^2 = d\theta^2 + \sin^2 \theta d\phi^2$ . That is to say, (15) and (18) describe the same frame in different coordinate bases.

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<sup>1</sup> The author is indebted to Rong-Xin Miao and the referee for their valuable comments on this point.

The above example sharpens the problem of what frame we should adopt to look for new solutions. It is interesting to notice that the spatial coordinates in metric (12) are Cartesian up to a conformal factor. This coordinate system is relatively more convenient for us to perform the Lorentz boost and the rotation of spatial axes locally. Fortunately, there is a close cousin of (12), namely the generalized Gullstrand-Painlevé metric

$$ds^2 = dt^2 - \delta_{ij} \left( \frac{\sqrt{\alpha} x^i}{r} dt + \frac{dx^i}{\beta} \right) \left( \frac{\sqrt{\alpha} x^j}{r} dt + \frac{dx^j}{\beta} \right), \quad (20)$$

where  $\alpha$  and  $\beta$  are functions of the radial coordinate  $r = (\delta_{ij} x^i x^j)^{1/2}$  for static solutions. In this coordinate system, every  $x^\mu$  fully spans  $(-\infty, +\infty)$ , and the spatial coordinates are conformally Cartesian [38, 39]. Through a coordinate transformation

$$t = \tilde{t} + \int \frac{\sqrt{\alpha}}{\beta(1-\alpha)} dr, \quad (21)$$

metric (20) can be rewritten in the more familiar form

$$\begin{aligned} ds^2 &= dt^2 - \left( \sqrt{\alpha} dt + \frac{dr}{\beta} \right)^2 - \frac{r^2}{\beta^2} d\Omega^2 \\ &= (1-\alpha) d\tilde{t}^2 - \frac{1}{\beta^2(1-\alpha)} dr^2 - \frac{r^2}{\beta^2} d\Omega^2 \end{aligned} \quad (22)$$

in spherical coordinates.

From the metric in Gullstrand-Painlevé coordinates, we directly read out one possible form of the tetrad fields

$$e_\mu^0 dx^\mu = dt, \quad e_\mu^i dx^\mu = \frac{\sqrt{\alpha} x^i}{r} dt + \frac{1}{\beta} dx^i. \quad (23)$$

This is the closest analogue of (13) for the solution we are interested in. It is also the starting point of our investigation. The impact of such a choice of frame is twofold. On the one hand, it simplifies our calculation significantly, otherwise the Lorentz transformation will induce six more variables and then the equations of motion would be too cumbersome to tackle. On the other hand, it restricts the validity of our solutions to a limited class of  $f(T)$  models, while the search for spherically symmetric static solutions in more general models remains a challenging task.

### III. HUNTING FOR SPHERICALLY SYMMETRIC STATIC SOLUTIONS

In the Maxwell term (8),  $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$  for an electromagnetic field. If the field is static and spherically symmetric, we can make use of  $U(1)$  gauge invariance to write it in the form  $A_\mu dx^\mu = \gamma(r) dt$ . The electric charge of our solution can be obtained by evaluating the integral

$$-\frac{1}{4\pi} \int \sqrt{-g} g^{tt} g^{rr} F_{tr} d\theta d\varphi = -\frac{r^2 \gamma_{,r}}{\beta} \quad (24)$$

in the limit  $r \rightarrow \infty$ .

Starting from tetrads of the form (23), we find the torsion scalar

$$T = -\frac{2}{r^2}[\alpha(\beta - r\beta_{,r})^2 + r\beta\alpha_{,r}(\beta - r\beta_{,r}) - r^2\beta_{,r}^2]. \quad (25)$$

The equations of motion are complicated, whose tedious form will not be shown here. We succeeded in casting them in simple forms under different situations. One situation is that with  $\alpha = 0$ . This will be studied in section IV. The other situation is to be investigated in section V, where  $\alpha \neq 0$ .

#### IV. $\alpha = 0, \beta \neq 0$

For physical solutions,  $\beta$  should be non-zero, but  $\alpha = 0$  is still allowed. In this situation, the torsion scalar simplifies as  $T = 2\beta_{,r}^2$ , and the equations of motion reduce to

$$r\beta \left[ \frac{\beta_{,r}}{r}(1 + f_{,T}) \right]_{,r} - \left( \frac{T}{2} - \frac{2\beta\beta_{,r}}{r} \right) (1 + f_{,T}) = 0, \quad (26)$$

$$\left( 2T - \frac{4\beta\beta_{,r}}{r} \right) (1 + f_{,T}) - (T + f) + 8\pi G\beta^2\gamma_{,r}^2 = 0, \quad (27)$$

$$r\beta \left[ \frac{\beta_{,r}}{r}(1 + f_{,T}) \right]_{,r} + 8\pi G\beta^2\gamma_{,r}^2 = 0, \quad (28)$$

$$\left( \frac{r^2\gamma_{,r}}{\beta} \right)_{,r} = 0. \quad (29)$$

Equation (26) can be integrated, resulting in the relation

$$\frac{r\beta_{,r}}{\beta}(1 + f_{,T}) = \frac{8\pi GQ^2}{r_\infty^2} \quad (30)$$

Here  $Q$  and  $r_\infty$  are constants, whose physical significance will be clear later. We will assume  $r_\infty > 0$  without loss of generality. Inserting this relation into (28), we get

$$\frac{Q^2}{r_\infty^2} \left( \frac{r\beta_{,r}}{\beta} - 2 \right) + r^2\gamma_{,r}^2 = 0 \quad (31)$$

because  $\beta \neq 0$ . This equation can be combined with (29) to give a second-order ordinary differential equation of  $\gamma$ ,

$$\frac{Q^2}{r_\infty^2}\gamma_{,rr} + r\gamma_{,r}^3 = 0, \quad (32)$$

which leads to

$$\gamma_{,r} = -\frac{Q}{r_\infty\sqrt{r^2 + r_0^2}} \quad (33)$$

with a nonnegative constant  $r_0$ . We have chosen a proper signature to ensure that  $Q$  is the electric charge given by (24). Now the  $\gamma_{,r}^2$  term can be eliminated in equation (31) and the equation becomes

$$\frac{Q^2}{r_\infty^2} \left( \frac{r\beta_{,r}}{\beta} - \frac{r^2 + 2r_0^2}{r^2 + r_0^2} \right) = 0. \quad (34)$$

When  $Q/r_\infty \neq 0$ , this is a first-order equation of  $\beta$ . We will continue our discussion in two situations:  $Q/r_\infty = 0$  in subsection IV A and  $Q/r_\infty \neq 0$  in subsection IV B.

### A. $Q/r_\infty = 0$

In the situation  $Q/r_\infty = 0$ , equation (33) indicates  $\gamma = 0$  up to a constant eliminable by  $U(1)$  gauge transformation. At the same time, the right hand side of equation (30) vanishes. To this equation there are two solutions.

One solution is a constant  $\beta$ , and we fix it to be  $\beta = 1$  by rescaling the spatial coordinates  $x^i$ . This solution, summarized as

$$\begin{aligned}\alpha &= 0, \\ \beta &= 1, \\ \gamma &= 0,\end{aligned}\tag{35}$$

describes a Minkowski spacetime. One may check the torsion scalar  $T = 0$  and the Ricci scalar  $R = 0$ . It is easy to also check that the constant solution satisfies (26), (28) and (29) automatically. But the solution is incompatible with equation (27) unless  $f(0) = 0$ . That does not mean the gravity theory must recede to the Einstein's theory to guarantee the existence of global Minkowski solution. In fact, it provides a sufficient but not necessary condition: the global Minkowski solution exists if function  $f(T)$  vanishes at point  $T = 0$  numerically. If this condition is violated, the Minkowski solution might still exist in frames other than (23).

The other solution to (30) is

$$1 + f_{,T} = 0.\tag{36}$$

Then equation (27) tells us

$$T + f = 0.\tag{37}$$

If  $T$  is not a constant, one should solve equations (36) and (37) analytically as differential equations. The only consistent solution to them is  $f = -T$  analytically. However, this solution is unphysical and should be ruled out, otherwise the theory of gravity is ill-defined by a null Lagrangian. If  $T$  is a constant, it will be enough to require these equations to hold numerically, not analytically. In other words, we do not have to solve them as differential equations. Remembering that  $T = 2\beta_{,r}^2$ , a constant  $T$  can be achieved by

$$\begin{aligned}\alpha &= 0, \\ \beta &= \frac{r}{r_\infty} + k, \\ \gamma &= 0.\end{aligned}\tag{38}$$

Here the constant  $k = 0$  or  $\pm 1$  after rescaling  $x^i$ . From (38) we work out the torsion scalar  $T = 2/r_\infty^2$  and the Ricci scalar

$$R = -\frac{2}{r_\infty^2} - \frac{8k}{r_\infty r}.\tag{39}$$

Clearly there is a curvature singularity at  $r = 0$  unless  $r_\infty$  is infinite or  $k = 0$ . So let us consider different choices of parameters.

1.  $r_\infty$  is infinite and  $k = 0$ . This solution gives  $\beta = 0$  and an ill-behaved metric.
2.  $r_\infty$  is infinite and  $k = \pm 1$ . We identify this solution as the Minkowski solution (35), but emphasize that equation (36) is not a necessary condition for the existence of Minkowski solution, as shown at the beginning of this subsection.

3.  $r_\infty$  is finite and  $k = 0$ . In terms of a new coordinate  $z = r_\infty \ln(r/r_\infty)$ , the metric from this solution takes the form

$$ds^2 = dt^2 - dz^2 - r_\infty^2 d\Omega^2. \quad (40)$$

This metric describes a spacetime of topology  $\mathbf{R}^1 \times \mathbf{R}^1 \times \mathbf{S}^2$ , where the  $\mathbf{S}^2$  sphere is of constant radius  $r_\infty$ .

4.  $r_\infty$  is finite and  $k = \pm 1$ . It is easy to see the curvature singularity at  $r = 0$  is a naked singularity in these solution.

### B. $Q/r_\infty \neq 0$

In situations with  $Q/r_\infty \neq 0$ , equation (34) is solved by

$$\beta = \frac{r^2}{r_\infty \sqrt{r^2 + r_0^2}}, \quad (41)$$

which immediately gives

$$T = \frac{2r^2(r^2 + 2r_0^2)^2}{r_\infty^2(r^2 + r_0^2)^3}. \quad (42)$$

One may use (24), (33) and (41) to prove that indeed  $Q$  is the electric charge.

Substituting equations (33), (34), (41) and (42) into (27) and (30), we obtain

$$\begin{aligned} T + f &= \frac{8\pi G Q^2 r^2 (r^2 + 4r_0^2)}{r_\infty^4 (r^2 + r_0^2)^2}, \\ 1 + f_{,T} &= \frac{8\pi G Q^2 (r^2 + r_0^2)}{r_\infty^2 (r^2 + 2r_0^2)}. \end{aligned} \quad (43)$$

Making use of (42), one may check the consistency condition  $(T + f)_{,r} = (1 + f_{,T})T_{,r}$  by straightforward calculations.

In order to work out more details, we should consider two possibilities.

1.  $r_0 = 0$ . For such a choice, we get the solution

$$\begin{aligned} \alpha &= 0, \\ \beta &= \frac{r}{r_\infty}, \\ \gamma &= -\frac{Q}{r_\infty} \ln \left( \frac{r}{r_\infty} \right). \end{aligned} \quad (44)$$

In the notation  $z = r_\infty \ln(r/r_\infty)$ , it gives metric (40) and  $\gamma = -Qz/r_\infty^2$ . Furthermore, equations (43) become

$$\begin{aligned} T + f &= \frac{8\pi G Q^2}{r_\infty^4}, \\ 1 + f_{,T} &= \frac{8\pi G Q^2}{r_\infty^2}, \end{aligned} \quad (45)$$

which should hold numerically for  $T = 2/r_\infty^2$ . Extrapolated to the limit  $Q = 0$ , this solution reproduces the regular  $\mathbf{R}^1 \times \mathbf{R}^1 \times \mathbf{S}^2$  solution presented in the previous subsection.



2.  $r_0 > 0$ . Then as a cubic equation of the variable  $r^2/r_0^2$ , equation (42) has one real root  $r^2/r_0^2 = u(T)$  and thus

$$\begin{aligned} T + f &= \frac{8\pi G Q^2 u(u+4)}{r_\infty^4 (u+1)^2}, \\ 1 + f_{,T} &= \frac{8\pi G Q^2 (u+1)}{r_\infty^2 (u+2)}. \end{aligned} \quad (46)$$

The metric is decided by the solution

$$\begin{aligned} \alpha &= 0, \\ \beta &= \frac{r^2}{r_\infty \sqrt{r^2 + r_0^2}}, \\ \gamma &= \frac{Q}{r_\infty} \operatorname{arcsinh} \left( \frac{r_\infty}{r_0} \right) - \frac{Q}{r_\infty} \operatorname{arcsinh} \left( \frac{r}{r_0} \right). \end{aligned} \quad (47)$$

In terms of coordinate

$$z = r_\infty \left[ \operatorname{arcsinh} \left( \frac{r}{r_0} \right) - \operatorname{arcsinh} \left( \frac{r_\infty}{r_0} \right) + \sqrt{1 + \frac{r_0^2}{r_\infty^2}} - \sqrt{1 + \frac{r_0^2}{r^2}} \right], \quad (48)$$

we write it as

$$ds^2 = d\tilde{t}^2 - dz^2 - \tilde{r}^2 d\Omega^2, \quad (49)$$

which describes a spacetime of topology  $\mathbf{R}^1 \times \mathbf{R}^1 \times \mathbf{S}^2$ . The radius of  $\mathbf{S}^2$  sphere is

$$\tilde{r} = r_\infty \sqrt{1 + \frac{r_0^2}{r^2}}. \quad (50)$$

In the region  $r \in (0, \infty)$ , both  $z$  and  $\tilde{r}$  are strict monotonic functions of  $r$ , so the radius  $\tilde{r}$  shrinks monotonically as  $z$  varies from  $-\infty$  to  $+\infty$ , as illustrated in figure 1. Extrapolating this solution to the limit  $r_0 = 0$ , we recover equations (40), (44), (45) as well as  $T = 2/r_\infty^2$  and  $z = r_\infty \ln(r/r_\infty)$ . Note the extrapolation from (46) to (45) is a little tricky: the special metric (40) exists as long as equations (45) hold numerically at the point  $T = 2/r_\infty^2$ , but (46) should stand analytically to guarantee the existence of the general metric (49).

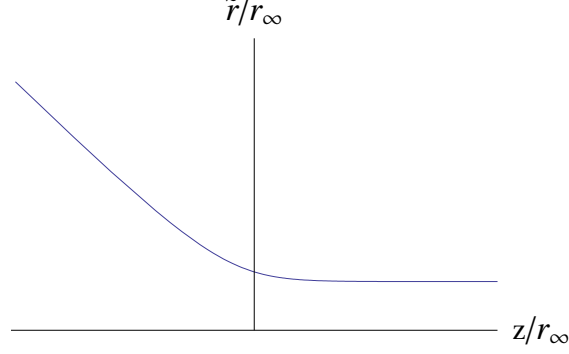


Figure 1: The radius  $\tilde{r}$  of  $\mathbf{S}^2$  (50) decreases monotonically as coordinate  $z$  (48) increases.

## V. $\alpha\beta \neq 0$

When  $\alpha\beta \neq 0$ , the equations of motion can be arranged in the form

$$(1 + f_{,T}) \left( \frac{r\beta_{,r}}{\beta} \right)_{,r} = 0, \quad (51)$$

$$f_{,TT}T_{,r} = 0, \quad (52)$$

$$\left( 2T - \frac{4\beta\beta_{,r}}{r} \right) (1 + f_{,T}) - (T + f) + 8\pi G\beta^2\gamma_{,r}^2 = 0, \quad (53)$$

$$\left[ 2\alpha \left( \frac{\beta}{r} - \beta_{,r} \right)^2 + \beta_{,r} \left( \frac{4\beta}{r} - \alpha_{,r}\beta - 2\beta_{,r} \right) - \alpha_{,rr}\beta^2 \right] (1 + f_{,T}) - 16\pi G\beta^2\gamma_{,r}^2 = 0, \quad (54)$$

$$\left( \frac{r^2\gamma_{,r}}{\beta} \right)_{,r} = 0. \quad (55)$$

The last equation is the non-vanishing component of Maxwell equations.

One special case is  $f_{,T} = -1$ . Keep in mind that  $\beta \neq 0$ , then equation (54) indicates  $\gamma = 0$  up to  $U(1)$  gauge transformation.

If  $f_{,T} \neq -1$ , equations(51) is solved by

$$\beta = \left( \frac{r}{r_\infty} \right)^p, \quad (56)$$

where we have chosen a positive signature for  $\beta$  without loss of generality. According to equation (55) we can write down the relation  $r^2\gamma_{,r}/\beta = -Q$  with a constant electric charge  $Q$ , as demonstrated by (24). This relation implies

$$\gamma = -\frac{Q}{r_\infty} \ln \left( \frac{r}{r_\infty} \right) \quad (57)$$

for  $p = 1$  or

$$\gamma = \frac{Q}{(1-p)r} \left( \frac{r}{r_\infty} \right)^p \quad (58)$$

for  $p \neq 1$  up to an eliminable constant.

We will study the case with  $f_{,T} = -1$  to some extent in subsection V A. The case with  $f_{,T} \neq -1$  will be exhaustively investigated, for  $p = 1$  in subsection V B, and for  $p \neq 1$  in subsections V C ( $T_{,r} \neq 0$ ) and V D ( $T = \text{const}$ ).

#### A. $f_{,T} = -1$

In this case, the equations of motion are further simplified,

$$\begin{aligned} 1 + f_{,T} &= 0, \\ f_{,TT} T_{,r} &= 0, \\ T + f &= 0. \end{aligned} \tag{59}$$

If  $T_{,r} \neq 0$ , equations (59) have only one solution  $f = -T$ , resulting in an unphysical null Lagrangian. So we turn to consider  $T = \text{const}$ .

The expression of  $T$  is given by (25). It is difficult to exhaust all of the solutions to  $T_{,r} = 0$ , but we found a particular solution

$$\begin{aligned} \beta &= \frac{r}{r_\infty}, \\ \gamma &= 0, \\ 1 + f_{,T} &= 0, \\ T + f &= 0, \end{aligned} \tag{60}$$

leaving  $\alpha$  unconstrained. The corresponding metric is

$$ds^2 = (1 - \alpha) d\tilde{t}^2 - \frac{r_\infty^2}{(1 - \alpha)r^2} dr^2 - r_\infty^2 d\Omega^2. \tag{61}$$

This metric generalizes the  $\mathbf{R}^1 \times \mathbf{R}^1 \times \mathbf{S}^2$  solution (40) of constant radius, but the constraints on Lagrangian are the same. For this solution, the torsion scalar  $T = 2/r_\infty^2$  and the Ricci scalar  $R = -(r^2 \alpha_{,rr} + r \alpha_{,r} + 2)/r_\infty^2$ .

Another particular solution to (59) is

$$\begin{aligned} \alpha &= \frac{\Lambda r^2}{3} + \frac{2GM}{r}, \\ \beta &= 1, \\ \gamma &= 0, \\ 1 + f_{,T} &= 0, \\ T + f &= 0 \end{aligned} \tag{62}$$

with  $T = -2\Lambda$ . Here  $\Lambda$  may be thought as the “cosmological constant” though this model does not include the Einstein’s gravity theory. Later in subsection V D we will see such a solution actually exist for more general models with any value of  $f_{,T}$ , which naturally incorporate the Einstein’s theory. Please refer to subsection V D for physical interpretations of this solution.

**B.**  $f_{,T} \neq -1, p = 1$

In the previous subsection, we have studied some solutions with  $f_{,T} = -1$ . From now on, we will consider the case of  $f_{,T} \neq -1$ . In this case, one special example is  $p = 1$ . Now the solution of  $\beta$  is given by (56) with  $p = 1$ , while the solution of  $\gamma$  is (57). Then one may directly check that  $T = 2/r_\infty^2$  using equation (25). Subsequently, most equations of motion are automatically satisfied except for (53) and (54), which demand

$$T + f = \frac{8\pi G Q^2}{r_\infty^4} \quad (63)$$

and

$$\alpha = \left[ 1 - \frac{8\pi G Q^2}{r_\infty^2(1 + f_{,T})} \right] \ln^2 \left( \frac{r}{r_\infty} \right) + \ln \left( \frac{r}{r_\infty} \right)^q + C \quad (64)$$

respectively. In the above solution,  $q$  and  $C$  are constants, and we have taken into account the fact that  $f_{,T}$  is independent of  $r$ . It is easy to show that the corresponding geometry contains an  $\mathbf{S}^2$  sphere of constant radius  $r_\infty$ . More interestingly, solution (44) and constraints (45) can be regenerated from this solution by requiring  $\alpha = 0$ .

In what follows, restricting to the case  $f_{,T} \neq -1$ , we will explore the possibility that  $p \neq 1$ . We will put our results in two subsections, according to whether the torsion scalar is a constant.

**C.**  $f_{,T} \neq -1, p \neq 1, T_{,r} \neq 0$

In this case  $T$  is not a constant, so the solution to equation (52) is  $T + f = (T - 2\Lambda)/\lambda$ . In order to give the correct value of Newtonian constant,  $\lambda = 1$  should be imposed on physically viable models. The viable model is simply the Einstein's gravity theory with a cosmological constant  $\Lambda$ . Although the model is not new, in this subsection we will briefly cover it for completeness.

Since  $f(T)$  is a constant, from equations (53) and (54) we obtain

$$\begin{aligned} T &= -2\Lambda + \frac{4p}{r^2} \left( \frac{r}{r_\infty} \right)^{2p} - \frac{8\pi G Q^2}{r^4} \left( \frac{r}{r_\infty} \right)^{4p}, \\ \alpha &= \frac{p(p-2)}{(p-1)^2} + \frac{C r^2}{(p-1)^2} \left( \frac{r_\infty}{r} \right)^{2p} + \frac{2GM}{(p-1)^2 r} \left( \frac{r}{r_\infty} \right)^p - \frac{4\pi G Q^2}{(p-1)^2 r^2} \left( \frac{r}{r_\infty} \right)^{2p}. \end{aligned} \quad (65)$$

Here  $M$  and  $C$  are constants of integration, while  $G$  is the Newtonian constant. Comparing the solution with (25), we can identify  $C = \Lambda/3$ .

With the redefinition  $\tilde{r} = r_\infty^p r^{1-p}$ , this solution can be reformed as

$$\begin{aligned} 1 - \alpha &= \frac{1}{(p-1)^2} \left( 1 - \frac{\Lambda \tilde{r}^2}{3} - \frac{2GM}{\tilde{r}} + \frac{4\pi G Q^2}{\tilde{r}^2} \right), \\ \beta^{-2} r^2 &= \tilde{r}^2, \\ \beta^{-2} dr^2 &= \frac{1}{(p-1)^2} d\tilde{r}^2, \\ \gamma &= \frac{Q}{(1-p)\tilde{r}}, \\ f &= -2\Lambda. \end{aligned} \quad (66)$$

After the time coordinate is rescaled as  $\tilde{t} \rightarrow (1 - p)\tilde{t}$ , solution (66) recovers the familiar Reissner-Nordström metric, which describes a charged black hole in de Sitter ( $\Lambda > 0$ ) or anti-de Sitter ( $\Lambda < 0$ ) or flat ( $\Lambda = 0$ ) spacetime. Other than the coordinate transformation in this way, we can also recover the metric directly from (56), (58) and (65) by setting  $p = 0$ . In this solution, the integration constants  $M$  and  $Q$  are mass and electric charge of the black hole respectively, while  $\Lambda$  is the cosmological constant.

**D.**  $f_{,T} \neq -1, p \neq 1, T_{,r} = 0$

Since  $p \neq 1$ , the expressions of  $\beta$  and  $\gamma$  are given by equations (56) and (58). We insert them into equation (53) and expand the resulted equation in powers of  $r$ ,

$$2Tf_{,T} - f + T - \frac{4p}{r^2}(1 + f_{,T}) \left(\frac{r}{r_\infty}\right)^{2p} + \frac{8\pi GQ^2}{r^4} \left(\frac{r}{r_\infty}\right)^{4p} = 0. \quad (67)$$

The first two terms on the left hand side are constants, because  $f$  and  $f_{,T}$  are constants when  $T$  takes a constant value. Remember also that  $f_{,T} \neq -1, p \neq 1$ , so this equation dictates  $p = 0, Q = 0$  and

$$2Tf_{,T} - f + T = 0. \quad (68)$$

Therefore, the consistent solution is restricted to

$$\beta = 1, \quad \gamma = 0. \quad (69)$$

Let us denote  $T = -2\Lambda$  as a constant. Then equation (25) can be solved as a differential equation of  $\alpha$  by

$$\alpha = \frac{\Lambda r^2}{3} + \frac{2GM}{r}. \quad (70)$$

One may check that equations (52) and (54) are satisfied automatically.

Extrapolating (68), (69) and (70) to the limit  $f_{,T} = -1$ , we reobtain equations (62), thus equation (68) holds more generally as an existence condition of this solution. The solution can be interpreted as a Schwarzschild black hole in de Sitter ( $\Lambda > 0$ ) or anti-de Sitter ( $\Lambda < 0$ ) or flat ( $\Lambda = 0$ ) spacetime. The integration constant  $M$  is related to the black hole mass, while  $\Lambda$  is a constant determined by the Lagrangian. Although  $\Lambda$  does not necessarily arise from a constant term in the Lagrangian, we will still call it cosmological constant for simplicity. It is straightforward to check that the Ricci scalar  $R = -4\Lambda$ .

We observe that torsion scalar  $T$  and equation (68) depend only on “cosmological constant”  $\Lambda$ , so the mass parameter  $M$  does not enter into Lagrangian parameters. This is consistent with our physical interpretations of the integration constants. Equation (68) puts a constraint on Lagrangian of viable models that admit such a solution. Since  $T$  is a constant, it is enough to require this equation to hold numerically, not analytically. In other words, we do not have to solve it as a differential equation.

In subsection IV A, we have found out a solution describing the Minkowski spacetime. In frame (23), the Minkowski solution exists in  $f(T)$  theories satisfying  $f(0) = 0$  numerically. Actually, as can be confirmed directly, this solution and its constraint on Lagrangian can be obtained as a limit  $M = 0, \Lambda = 0$  of the Schwarzschild solution in this subsection.

## VI. RECONSTRUCTION OF LAGRANGIAN

The results in sections IV and V look scattering. In the current section let us review them briefly by collecting the existence conditions of some solutions. In other words, we will review the conditions  $f(T)$  should meet if certain solutions exist in frame (23). As will be shown, some of the results can be unified in the same form, although they were derived under different assumptions. As a convention, when we say an existence condition should hold “analytically”, we mean the equations should stand for any value of  $T$ , solved as a differential equation analytically. Otherwise, the word “numerically” is used to mean that the equations are only to hold for a specified value of  $T$ . We should stress that all of the existence conditions here apply only to the frame (23). But we do not claim anything about other frames. Particularly, when the conditions are violated, these solutions may appear in a different frame.

### A. Reissner-Nordström-(anti-)de Sitter Solution

We have investigated this solution in subsection V C. Using the notations of (23), this solution is

$$\begin{aligned}\alpha &= \frac{p(p-2)}{(p-1)^2} + \frac{\Lambda r^2}{3(p-1)^2} \left(\frac{r_\infty}{r}\right)^{2p} + \frac{2GM}{(p-1)^2 r} \left(\frac{r}{r_\infty}\right)^p - \frac{4\pi G Q^2}{(p-1)^2 r^2} \left(\frac{r}{r_\infty}\right)^{2p}, \\ \beta &= \left(\frac{r}{r_\infty}\right)^p, \\ \gamma &= \frac{Q}{(1-p)r} \left(\frac{r}{r_\infty}\right)^p\end{aligned}\tag{71}$$

with  $p \neq 1$ . Solution (71) exists in frame (23) if

$$f = -2\Lambda\tag{72}$$

analytically, which is equivalent to the Einstein’s theory with a cosmological constant  $\Lambda$ . We achieved at this condition in subsection V C under the assumption  $f_{,T} \neq -1$ . Nevertheless, by combining the fact that  $T_{,r} \neq 0$  for this solution and the discussions at the beginning of subsection V A, we can see the condition is robust even without any assumption on  $f_{,T}$ .

### B. Schwarzschild-(anti-)de Sitter Solution

The solution is

$$\begin{aligned}\alpha &= \frac{\Lambda r^2}{3} + \frac{2GM}{r}, \\ \beta &= 1, \\ \gamma &= 0\end{aligned}\tag{73}$$

in frame (23). The existence condition of solution (73) is

$$2Tf_{,T} - f + T = 0\tag{74}$$

numerically with a constant  $T = -2\Lambda$ . We can also consider (74) as the existence condition of (anti-)de Sitter or Minkowski solution in that frame, depending on the value of  $\Lambda$ . This solution has been studied in subsection V D, which stands on a general ground. The special case with  $f_{,T} = -1$  was explored in subsection V A, while the limit of  $M = 0$ ,  $\Lambda = 0$  was investigated in subsection IV A.

### C. Charged Solution with a Sphere of Constant Radius

This solution is

$$\begin{aligned}\alpha &= \left[1 - \frac{8\pi G Q^2}{r_\infty^2(1 + f_{,T})}\right] \ln^2\left(\frac{r}{r_\infty}\right) + \ln\left(\frac{r}{r_\infty}\right)^q + C, \\ \beta &= \frac{r}{r_\infty}, \\ \gamma &= -\frac{Q}{r_\infty} \ln\left(\frac{r}{r_\infty}\right)\end{aligned}\tag{75}$$

in frame (23). Its existence condition is

$$T + f = \frac{8\pi G Q^2}{r_\infty^4}\tag{76}$$

numerically, where  $T = 2/r_\infty^2$ . The above solution was obtained in subsection V B, but the result can be extrapolated to the special case  $\alpha = 0$ , namely the  $\mathbf{R}^1 \times \mathbf{R}^1 \times \mathbf{S}^2$  solution in subsection IV B. In that limit, there is an additional existence condition

$$1 + f_{,T} = \frac{8\pi G Q^2}{r_\infty^2}.\tag{77}$$

This condition can be successfully recovered from the requirement  $\alpha = 0$ . Further taking limit  $Q/r_\infty = 0$ , we find the neutral  $\mathbf{R}^1 \times \mathbf{R}^1 \times \mathbf{S}^2$  solution discussed in subsection IV A. The neutral solution can be extended in another way, as to be elucidated in the coming subsection.

### D. Neutral Solution with a Sphere of Constant Radius

In frame (23), the solution is

$$\begin{aligned}\beta &= \frac{r}{r_\infty}, \\ \gamma &= 0,\end{aligned}\tag{78}$$

leaving  $\alpha$  unconstrained. This solution exists in frame (23) if the Lagrangian satisfies conditions

$$\begin{aligned}1 + f_{,T} &= 0, \\ T + f &= 0\end{aligned}\tag{79}$$

numerically at  $T = 2/r_\infty^2$ . We have investigated this solution in subsections IV A ( $\alpha = 0$ ) and V A ( $\alpha \neq 0$ ). Both solutions (75) and (78) have a product space  $\mathbf{S}^2$  of constant radius  $r_\infty$ .

### E. Solution with a Sphere of Variable Radius

As has been discussed in subsection IV B, the solution

$$\begin{aligned}\alpha &= 0, \\ \beta &= \frac{r^2}{r_\infty \sqrt{r^2 + r_0^2}}, \\ \gamma &= \frac{Q}{r_\infty} \operatorname{arcsinh} \left( \frac{r_\infty}{r_0} \right) - \frac{Q}{r_\infty} \operatorname{arcsinh} \left( \frac{r}{r_0} \right).\end{aligned}\tag{80}$$

has a product space  $\mathbf{S}^2$  of variable radius (50). The existence condition of this solution in frame (23) can be cast in the form

$$T + f = \frac{8\pi G Q^2 u(u+4)}{r_\infty^4 (u+1)^2},\tag{81}$$

which is expected to hold analytically. In condition (81), function  $u(T)$  is implicitly given by the real root of cubic equation

$$T = \frac{2u(u+2)^2}{r_\infty^2 (u+1)^3}.\tag{82}$$

Note the second equation of (46) can be derived from the first one and equation (82).

### F. Example of Lagrangian Reconstruction

The existence conditions summarized above are useful. On the one hand, given the Lagrangian of  $f(T)$  model, we can quickly judge the existence of these solutions in frame (23). On the other hand, we can use them to reconstruct the Lagrangian admitting certain solutions. Interestingly, some of the existence conditions do not contradict with each other, so the corresponding solutions could coexist in the same Lagrangian.

As a simplified example, let us construct a model that admits a de Sitter solution, an anti-de Sitter solution and a Minkowski solution at the same time. According to the existence condition in subsection VIB, this can be realized by designing a function  $f(T)$  obeying

$$2Tf_{,T} - f + T \propto T(T + 2\Lambda_1)(T + 2\Lambda_2).\tag{83}$$

Assuming the polynomial form of  $f(T)$ , we build a model

$$f = \tilde{G}^2 \left[ T^3 + \frac{10}{3}(\Lambda_1 + \Lambda_2)T^2 + 20\Lambda_1\Lambda_2T \right] - T,\tag{84}$$

where  $\tilde{G}$  is a constant with the dimension of  $\text{mass}^{-2}$ . When  $\Lambda_1\Lambda_2 < 0$ , this model has the required multiple solutions indeed.



## VII. DISCUSSION

Based on a special frame (23), we found some static solutions with spherical symmetry in  $f(T)$  gravity theories, where the Maxwell term was taken into consideration. The solutions and their existence conditions were summarized in section VI. But there are several problems unsolved.

First, the tetrads (13) for FLRW spacetime are widely used in the literature on  $f(T)$  cosmology. The frame (23) we employed is the close cousin of (13). However, as we have shown in section II, the reconstruction of Lagrangian is very sensitive to the choice of frame, because the torsion scalar  $T$  is not locally Lorentz invariant. This poses the pressing problem: why are the frames like (13) so special and what shall we interpret the local Lorentz transformations in general  $f(T)$  theories? As an up-to-date reference, [40] provides a nice answer to this problem for FRLW spacetime, which is further illustrated in [41].

Second, our ansatz (23) of tetrads simplifies the computation greatly, but it also limits the validity of our results. This means we have only found some spherically symmetric static solutions for certain special  $f(T)$  models. It is still an open problem to get such solutions for more general models in other frames. That would involve six more variables related to the local Lorentz transformation, leading to equations of motion hard to solve.

Third, in subsection V A we only laid out some particular solutions to  $T_{,r} = 0$ . Investigation is still needed to exhaust all of its solutions.

Fourth, since the existence of black solutions has been confirmed in this paper, the thermodynamics of gravity is to be studied in  $f(T)$  theories.

## Acknowledgments

The author would like to thank Qinyan Tan for encouragement and support.

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